Marginal stability of the generalized random Lotka-Volterra model: logistic growth case and beyond

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Based on:

A. Altieri, S. Franz, Rapid Comm. Phys. Rev. E **99** (1) (2019)

A. Altieri, F. Roy, C. Cammarota, G. Biroli, Phys. Rev. Lett. **126**, 258301 (2021)

A. Altieri, G. Biroli, SciPost Physics **12**, 013 (2022)

G. Garcia Lorenzana, A. Altieri, Phys. Rev. E **105** (2022)

EverEvol - Population dynamics: from rare events to evolution

Grenoble - December 1st, 2022

MSC

Université Paris Cité

For Women in Science

FONDATION L'ORÉAL

Understanding the mechanisms that shape the diversity, stability and functioning
in ecological & biological systems is a **timely** and **challenging** problem.

- How does **diversity** affect the evolution of the other species? What is the diversity precisely?

- How does an ecosystem respond to external (pulse or probe) **perturbations**?

- Is a **multiple equilibria regime** possible for the same ecosystem?

- Can a population undergo **chaotic dynamics** and **avalanches** (extinction cascades)?

- How to describe cooperative **pattern formation** in complex communities?
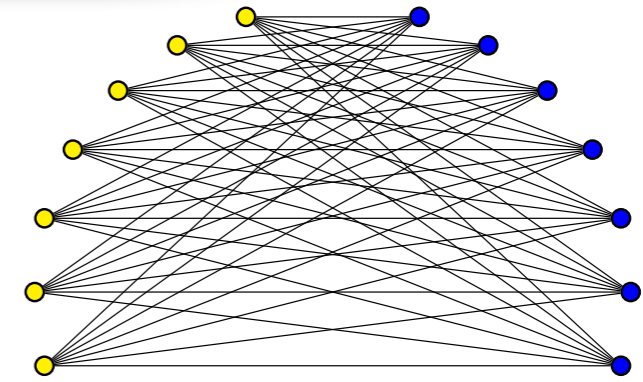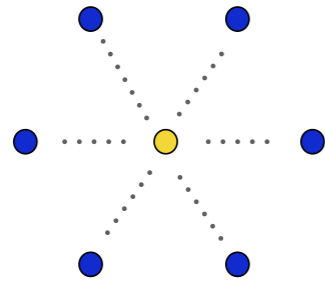
C. Chen, S. Liu, X. Shi, *et* al., Nature 542 (2017)
R.J. Schmitt, S.J. Holbrook *et al,* PNAS 116 (10) 4372 (2019)
J. Sirota, B. Baiser, N.J. Gotelli, A.M. Ellison, PNAS 110, 7742 (2013)
E. Beninca, J. Huisman *et al.,* Nature 451: 822 (2008)
E. Beninca, K.D. Johnk *et al.,* Ecol Lett 12: 1367 (2009).

From a few to many interacting components

STATISTICAL MECHANICS

Emergent mechanisms and collective behaviour $\longrightarrow$ PHASE TRANSITIONS

R.V. Solé, D. Alonso, A. McKane, Phil. Trans. Roy. Soc. B 357 (2002)
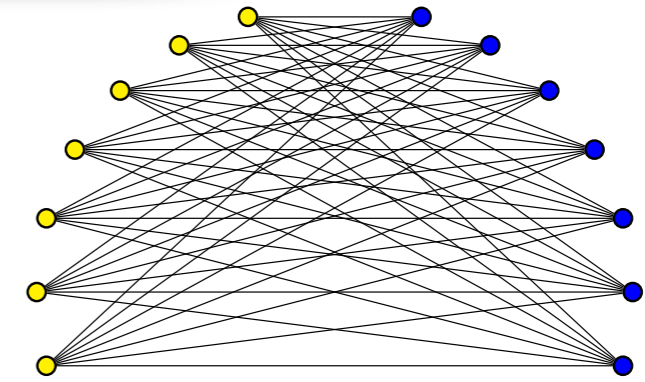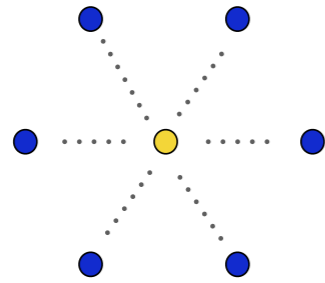
D. A. Kessler, N. M. Shnerb, Phys. Rev. E 91 (2015)

G. Bunin, Phys. Rev. E 95 (2017)

G. Biroli, G. Bunin, C. Cammarota, New Journal of Physics 20 (2018)

M. Tikhonov, R. Monasson, Phys. Rev. Lett. 118 (2017)

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Need for a general framework bridging the gap between models and experiments

MacArthur's model
(purely competitive)

Lotka-Volterra model
(competitive or cooperative)

- A. Altieri, S. Franz, Phys. Rev. E **99** (1) (2019)

- A. Altieri, F. Roy *et al.*, Phys. Rev. Lett.**126** (2021)
- A. Altieri, G. Biroli, SciPost Physics **12** (2022)
- G. Garcia Lorenzana, A. Altieri, Phys. Rev. E **105** (2022)

The random Lotka-Volterra model for species rich ecosystems

Dynamical equations for the relative species abundances $N_i \geq 0$, with $i = 1, ..., S$

$$\frac{dN_i}{dt} = -N_i \left[\nabla_{N_i} V_i(N_i) + \sum_{j(j \neq i)} \alpha_{ij} N_j \right] + \underbrace{\sqrt{N_i} \eta_i(t)}_{\text{noise}} + \lambda_i$$

noise   immigration rate

$$V_i(N_i) = -\rho_i \left(K_i N_i - \frac{N_i^2}{2} \right) \qquad \text{with} \quad \rho_i = r_i/K_i \text{ (growth rate/carrying capacity).}$$

Main assumptions:

- Well-mixed community (no space dependence)

- Demographic fluctuations modelled by Gaussian white noise with $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t) \eta_j(t') \rangle = 2T \delta_{ij} \delta(t - t')$ (within Ito's prescription).

- Complex behaviour described by <u>random interactions</u> α_{ij} with $\langle \alpha_{ij} \rangle = \mu/S$, $\langle \alpha_{ij}^2 \rangle_c = \sigma^2/S$, $\langle \alpha_{ij} \alpha_{ji} \rangle_c = \gamma \langle \alpha_{ij}^2 \rangle_c$

R. May, *Stability and Complexity in Model Ecosystem,* Princeton University Press (1973)

Fisher, Mehta PNAS 111 (2014)

M. Barbier, J.F. Arnoldi, G. Bunin, M. Loreau, PNAS 115 (2018)

G. Bunin, Phys. Rev. E 95 (2017)

L. Sidhom, T. Galla, Phys. Rev. E 101(3) (2020).

The dynamical equation admits an invariant probability distribution with:

$$P(\{N_i\}) = \exp\left(-\frac{H(\{N_i\})}{T}\right)$$

$$H = -\sum_{i=1}^{S}\left(N_i - \frac{N_i^2}{2}\right) + \sum_{i<j=1}^{S}\alpha_{ij}N_iN_j + \sum_{i=1}^{S}[T\ln N_i - \ln\theta(N_i - \lambda)]$$
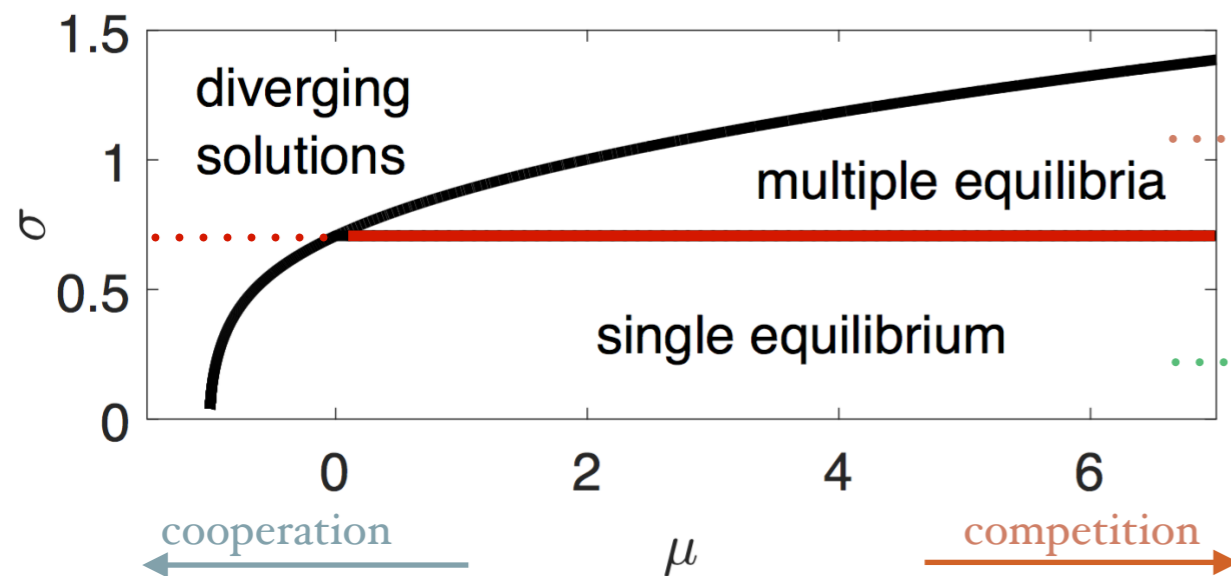
Demographic noise   Reflective wall

O.S. Venturelli, A. V. Carr, G. Fisher et al., Molecular systems biology, 14 (2018)
C. Ratzke, J. Barrere, J. Gore, Nature ecology & evolution 4.3 (2020)

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Phase diagram at T=0

diverging solutions

multiple equilibria

single equilibrium

cooperation

competition

μ

σ

- What are the typical features of these equilibria?
- How many equilibria?

- Stability decreases upon increasing heterogeneity

  (May's conjecture through RMT)

G. Bunin, Phys. Rev. E **95** (2017); G. Biroli, G. Bunin, C. Cammarota, New Journal of Physics **20** (2018).
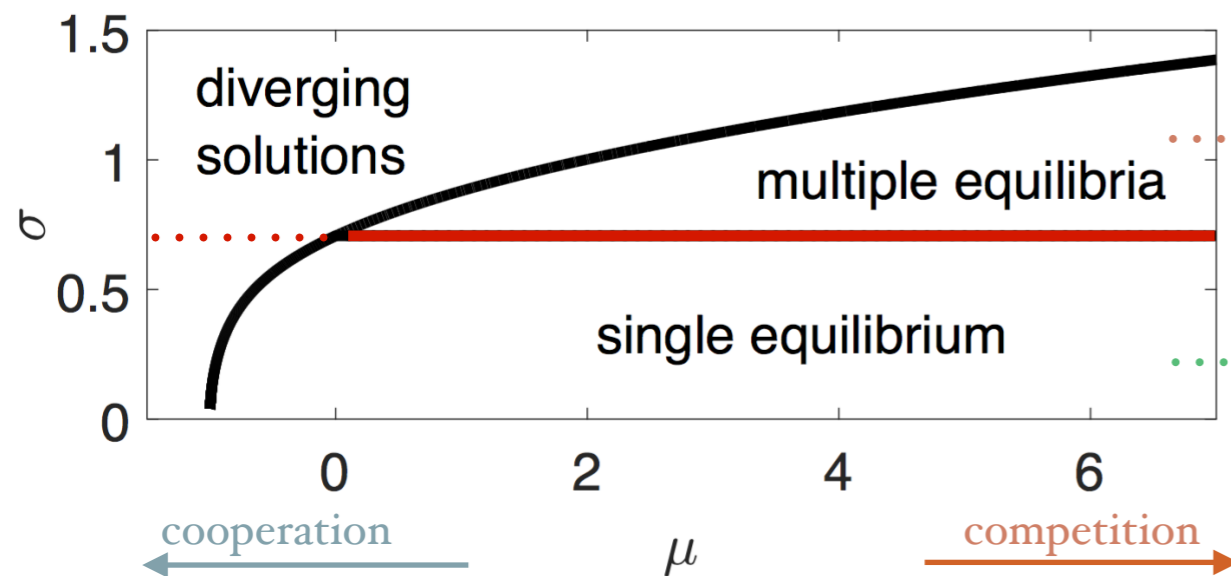
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G. Bunin, Phys. Rev. E **95** (2017); G. Biroli, G. Bunin, C. Cammarota, New Journal of Physics **20** (2018).

What is the **effect of the demographic noise**?

Interplay between **stochasticity and immigration**?

O.S. Venturelli, A. V. Carr, G. Fisher et al., Molecular systems biology, 14 (2018)
C. Ratzke, J. Barrere, J. Gore, Nature ecology & evolution 4.3 (2020)

To evaluate the (disordered) free energy, an average over random couplings must be performed.

$$-\beta n F = \lim_{n \to 0} \ln \overline{Z^n} = \lim_{n \to 0} \ln \int \prod_{i,(ij)} dN_i^a \, d\alpha_{ij} \exp \left[-\sum_{(ij)} \frac{\alpha_{ij} - \mu/S}{2\sigma^2/S} - \beta H(\{N_i^a\}) \right]$$
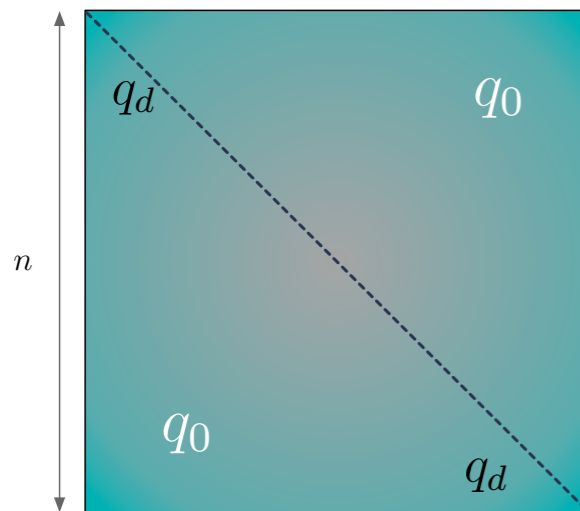
- $Q_{ab} = \dfrac{1}{S} \sum_{i=1}^{S} N_i^a N_i^b$

$\longrightarrow$

$$-\beta n F = \lim_{n \to 0} \ln \overline{Z^n} = \lim_{n \to 0} \ln \left[\int \prod_{ab} dQ_{ab} dQ_{aa} dH_a \, e^{S\mathcal{A}(Q_{ab}, Q_{aa}, H_a)} \right]$$

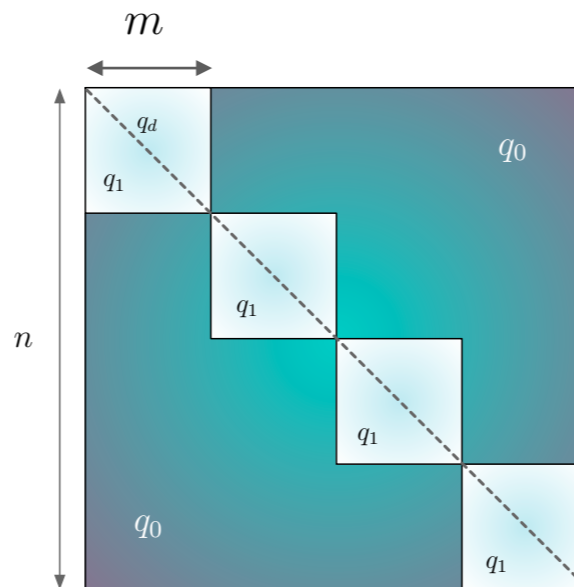- $H_a = \dfrac{1}{S} \sum_{i=1}^{S} N_i^a$

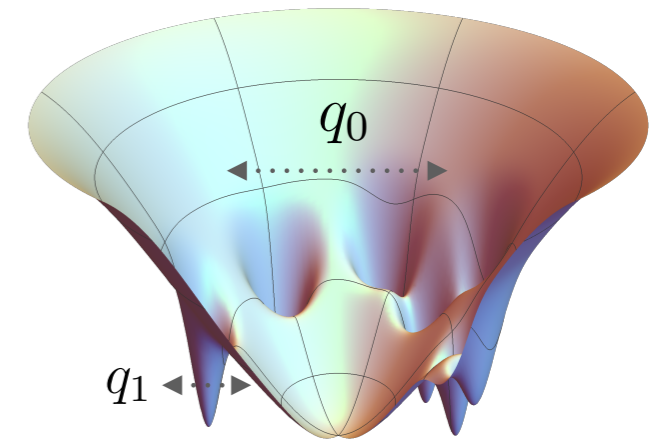
RS Ansatz

$$Q_{aa} = q_d \quad if \quad a = b$$
$$Q_{ab} = q_0 \quad if \quad a \neq b$$
$$H_a = h \qquad \quad \forall a$$

1RSB Ansatz

- $q_0 = \dfrac{1}{S} \sum_i \langle N_i \rangle_\alpha \langle N_i \rangle_\beta$

- $q_1 = \dfrac{1}{S} \sum_i \langle N_i \rangle_\alpha^2$
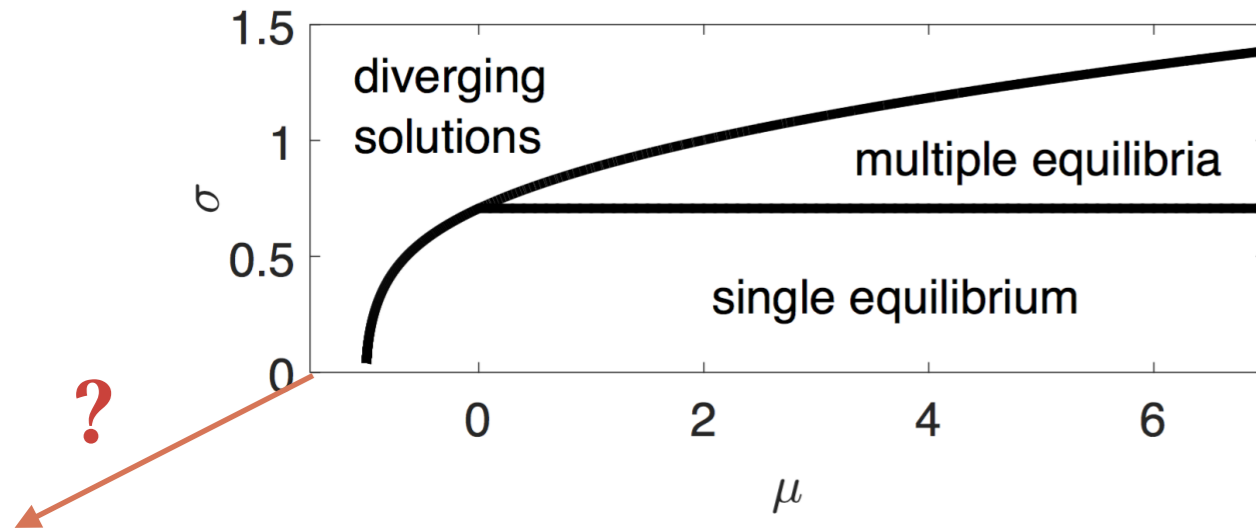
Landscape structure

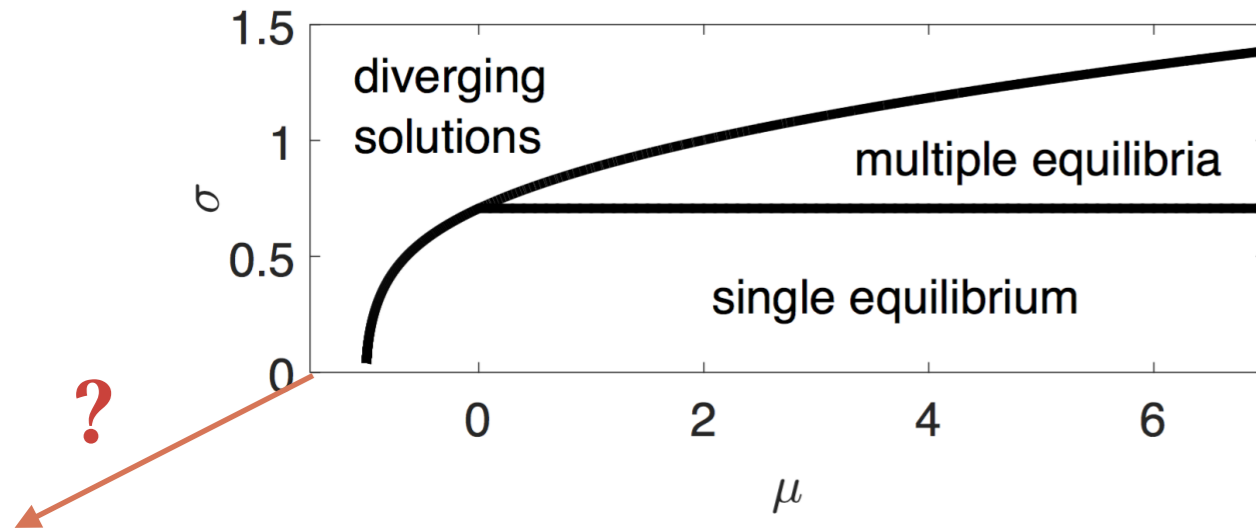size of the largest basin

size of the innermost basin

Goal: characterizing emergent collective behaviours in terms of disordered glass-like phases.

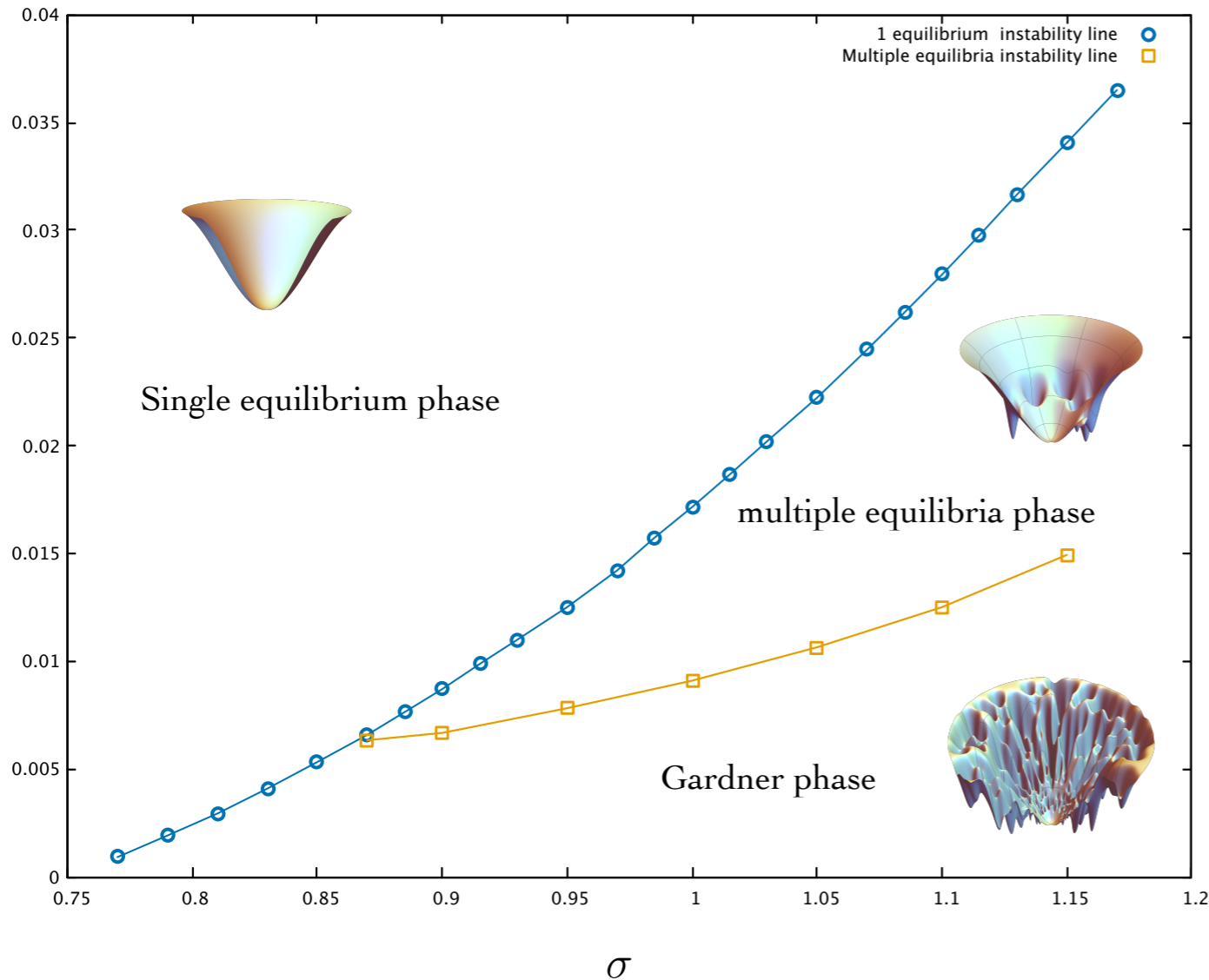<u>Goal</u>: characterizing emergent collective behaviours in terms of disordered glass-like phases.

Analysis of the harmonic fluctuations of the free energy:

$$\mathcal{M}_{abcd} = -\frac{\partial^2 \mathcal{A}}{\partial Q_{ab}\partial Q_{cd}} = \beta^2 \rho^2 \sigma^2 \left[\delta_{(ab),(cd)} - (\beta^2\rho^2\sigma^2)\overline{\langle N^a N^b, N^c N^d \rangle_c} \right]$$

Whose smallest eigenvalue in the replica formalism reads: $\lambda_{\mathrm{R}} = (\beta\rho\sigma)^2 \left[1 - (\beta\rho\sigma)^2 \overline{(\langle N^2 \rangle - \langle N \rangle^2)^2} \right]$

<u>Goal</u>: characterizing emergent collective behaviours in terms of disordered glass-like phases.

- No sensitive dependence on the average interaction parameter.

- Exponential number of locally stable states

$$\Sigma(f) \equiv \lim_{S \to \infty} \frac{1}{S} \overline{\log \mathcal{N}(f)} \quad \dashrightarrow \quad \mathcal{N}_{\text{eq}} \sim \exp S$$

* <u>Stability landscape:</u>
  Beisner et al., Front Ecol. Environ. (2003);
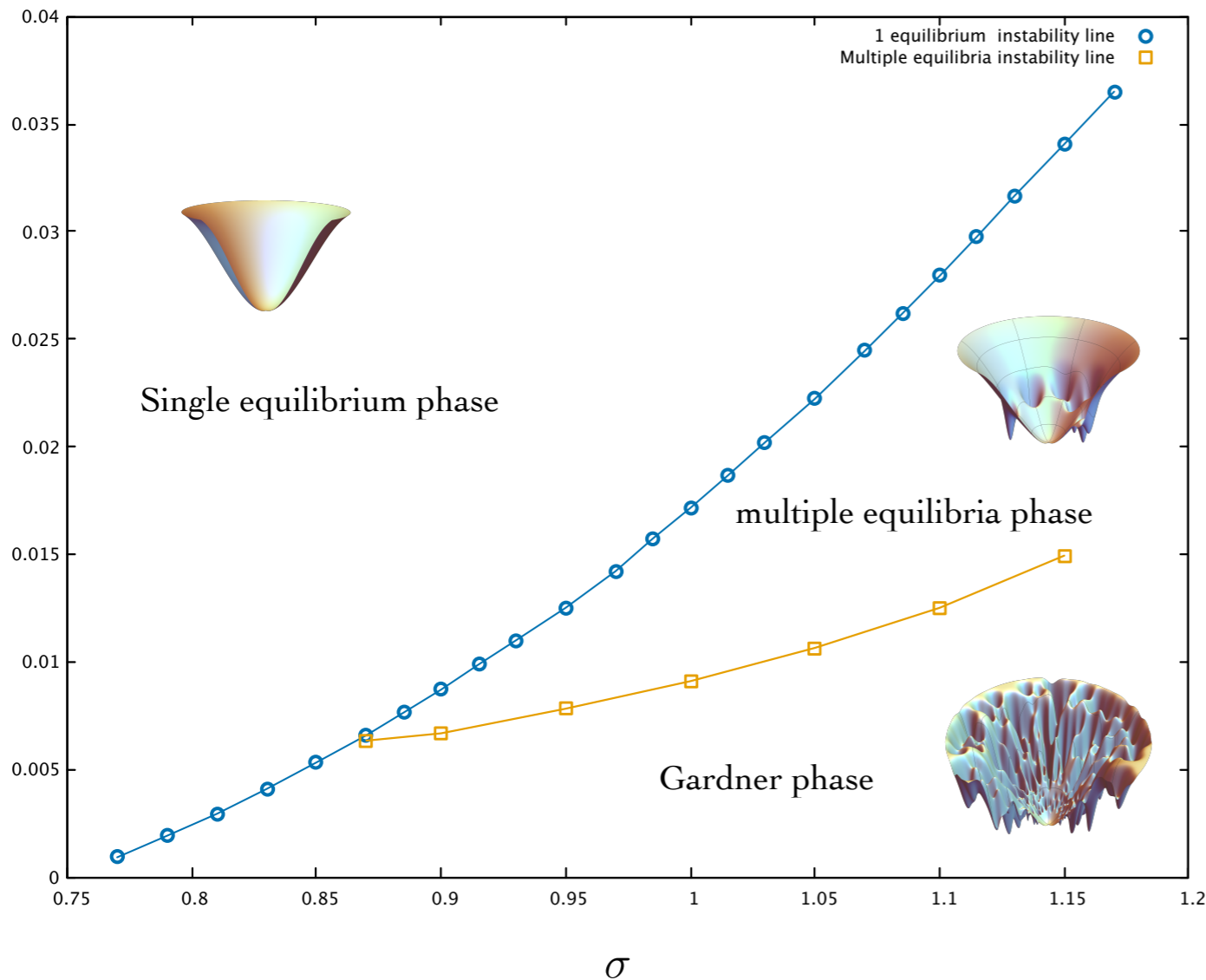 Scheffer and Carpenter, Trends in Ecology and Evolution (2003)

* <u>Ecological Resilience:</u>
  Botton et al., Critical reviews in microbiology 32.2 (2006)

· **Recent works by J. Gore's group** (2021);

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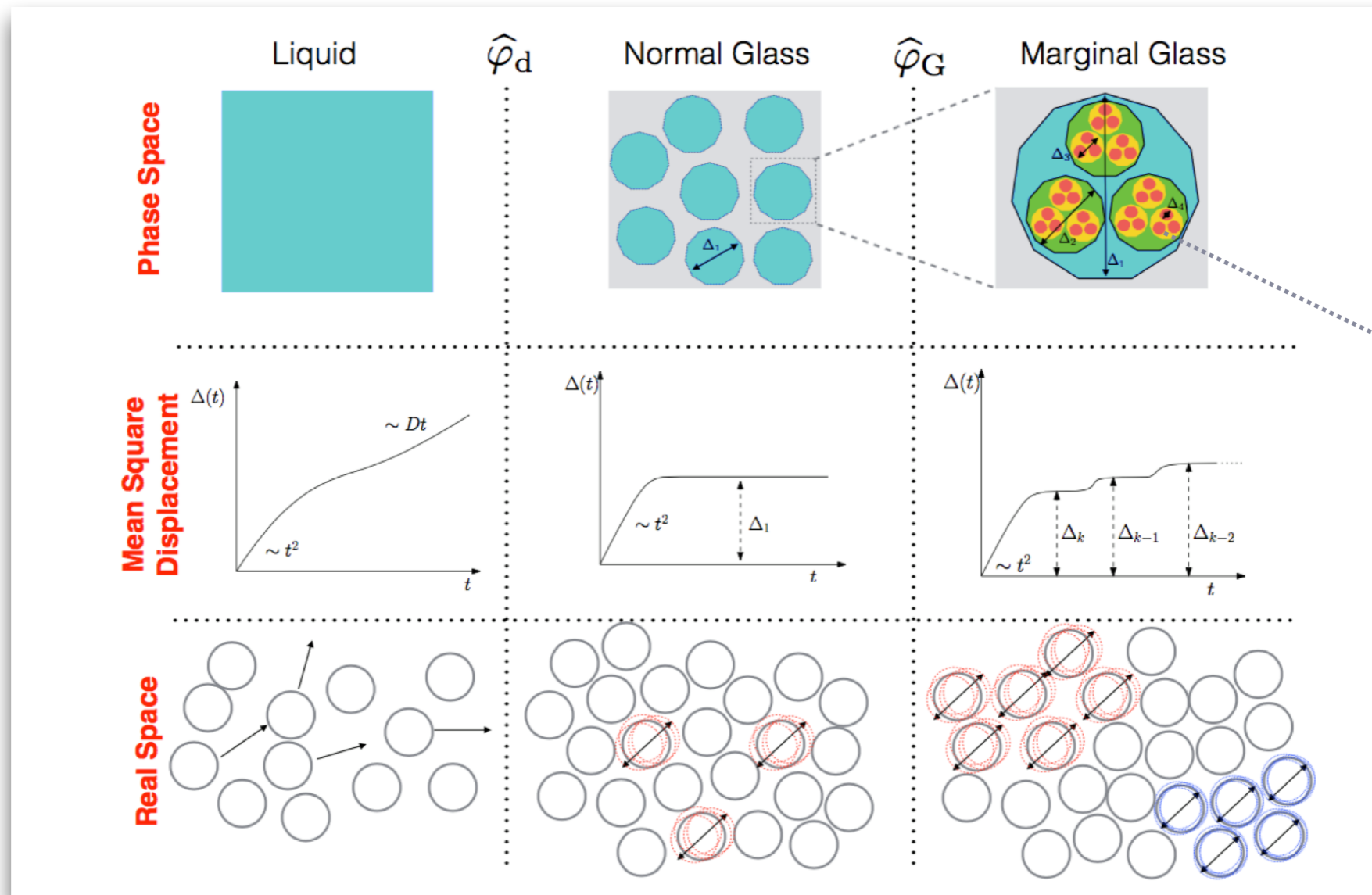
* <u>Ecological Resilience:</u>
  Botton et al., Critical reviews in microbiology 32.2 (2006)

· **Recent works by J. Gore's group** (2021);

- At very low demographic noise a **marginally stable amorphous phase**

First evidence of a **Gardner phase** akin to low-temperature glassy system phenomenology.

A. Altieri, F. Roy *et al.*, Phys. Rev. Lett. **126**, 258301 (2021)   13

P. Charbonneau, J. Kurchan, G. Parisi, P. Urbani. F. Zamponi, Annu. Rev. Condens. Matter Phys. **8** (2017)

Liquid phase: single smooth basin, reflecting the unbroken symmetry (ergodic phase);

Stable glass phase: many smooth and distinct basins characterizing the landscape

Marginal glass: infinitely broken phase ⟶ each basin breaks up into many sub-basins (hierarchically organised).
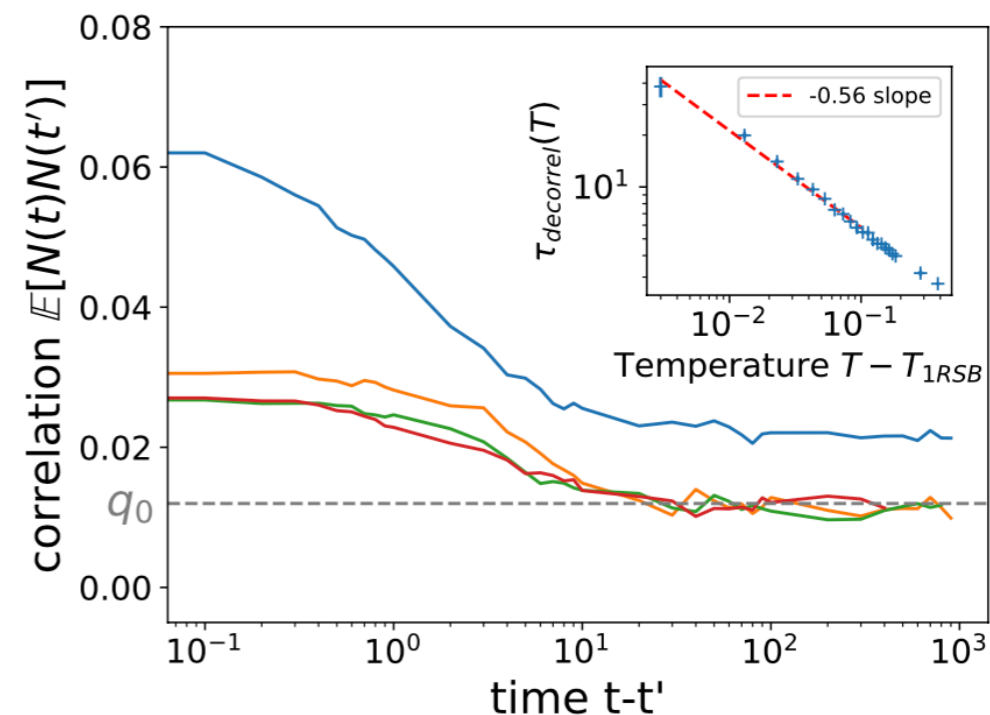
Dynamical mean-field theory (DMFT)

Dynamical correlation functions

○ **Single equilibrium phase**

$$\mathbb{E}[N(t)N(t')] = \frac{1}{SN_{sample}} \sum_{i=1}^{S} \sum_{r=1}^{N_{sample}} N_i^r(t) N_i^r(t')$$

$$\forall t \geq t' > t_{wait} \quad \mathbb{E}[N(t)N(t')] = C(t,t') \simeq C(t-t')$$
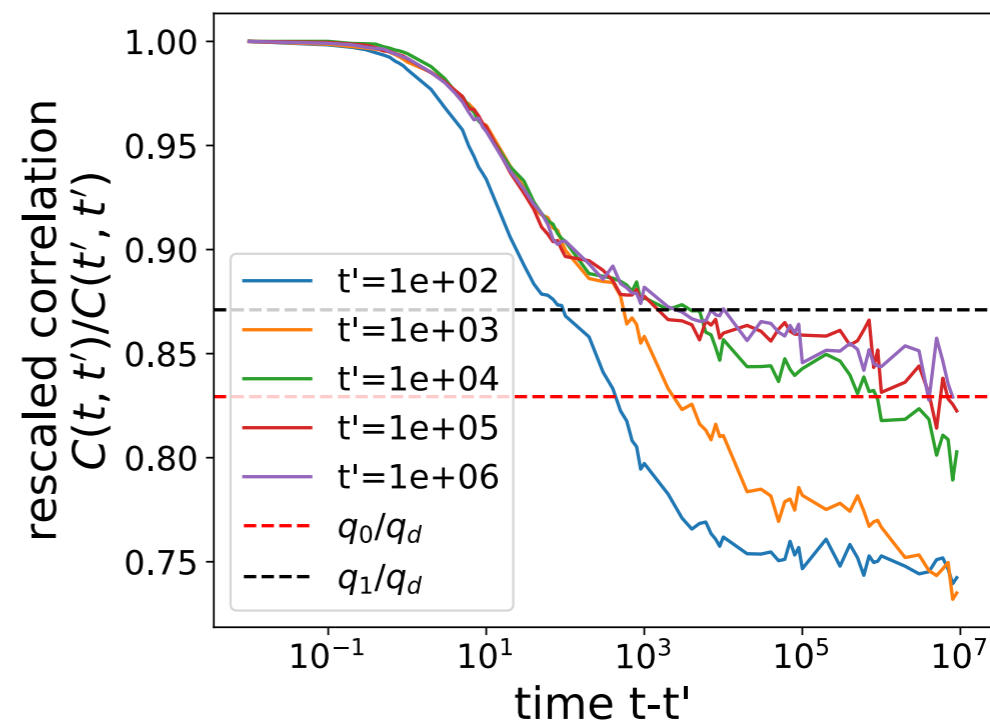
Time translationally invariant (TTI) regime

○ **Multiple equilibria phase**

Dramatic slowing down of the dynamics:

It never sits down in any of these equilibria wandering across the most numerous and marginally stable directions.

Aging dynamics as in mean-field spin-glasses

L. Cugliandolo, J. Kurchan, PRL (1993)

16

Replicator equations of great interest in: population genetics, analysis of Nash equilibria,

evolutionary game theory, and complex economic systems.

J. Hofbauer, J. Nonlinear Anal. 5, 1003 (1981)
S. Diederich, M. Opper, Phys. Rev. A 39 (1989)
T. Galla, J. D. Farmer, PNAS 110 (2013)
G. W.A. Constable, A. J. McKane, Phys. Rev. E 96 (2017)

$$H = \sum_i V_i(N_i) + \sum_{ij} \alpha_{ij} N_i N_j \qquad \mathcal{H}_{\mathrm{R}} = \sum_{i<j=1}^{S} J_{ij} x_i x_j - a \sum_{i=1}^{S} x_i^2 \qquad \sum_i x_i = S$$

Additional constraint

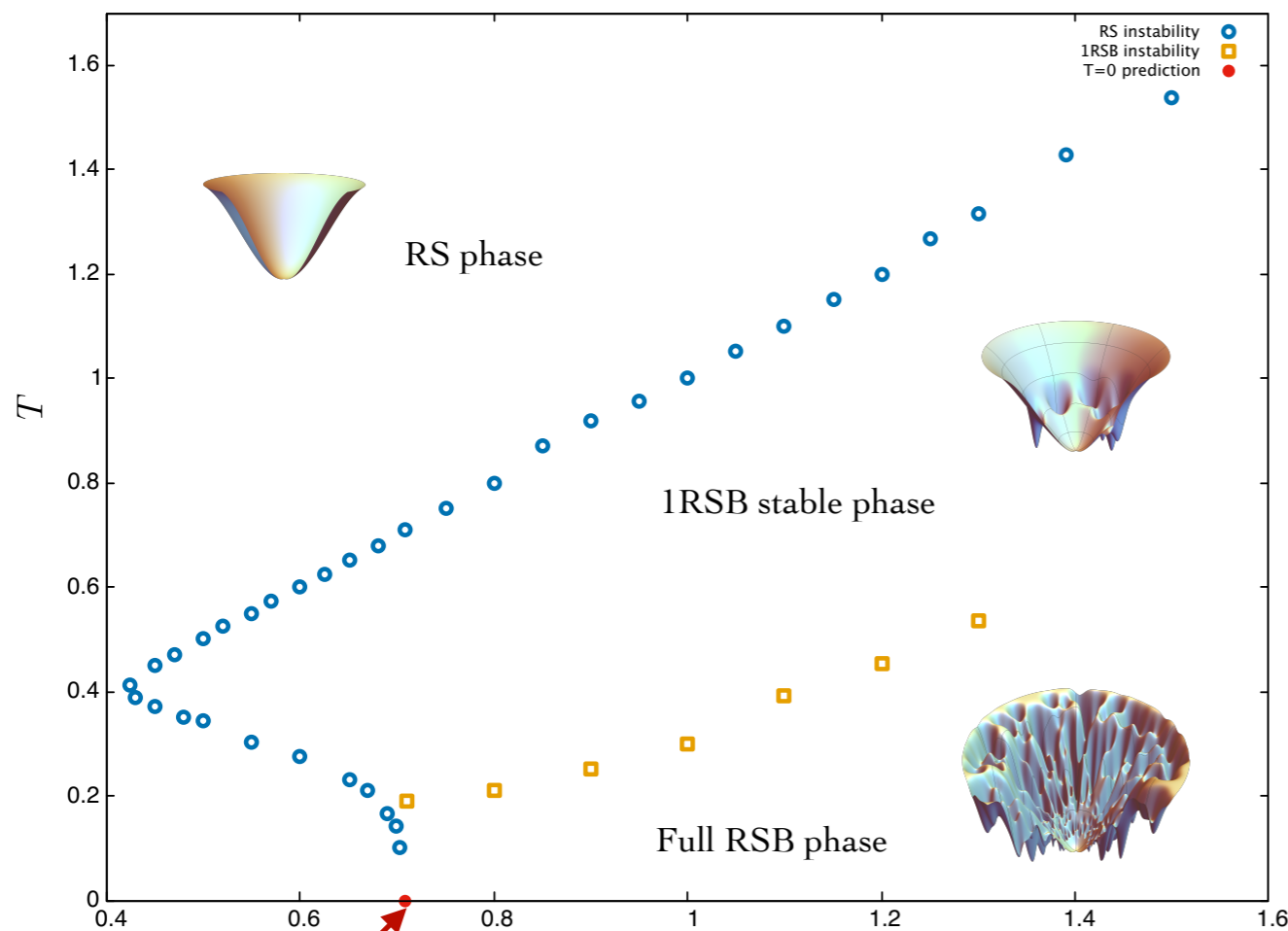P. Biscari, G. Parisi, J. Phys. A 28 (1995)

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P. Biscari, G. Parisi, J. Phys. A 28 (1995)

We established an exact mapping between the two:

$$\left[\rho - \rho^2 \sigma^2 \beta (q_d - q_0)\right] \Longleftrightarrow -\beta J(a/J - q)$$

$$\left(\rho \mu h - z\rho\sigma\sqrt{q_0} - \rho\right) \Longleftrightarrow -2z\sqrt{\tilde{t}} - \tilde{\gamma}$$

Zero-demographic noise critical point.

? Effect of introducing **non-symmetric interactions/non-conservative forces**?

18

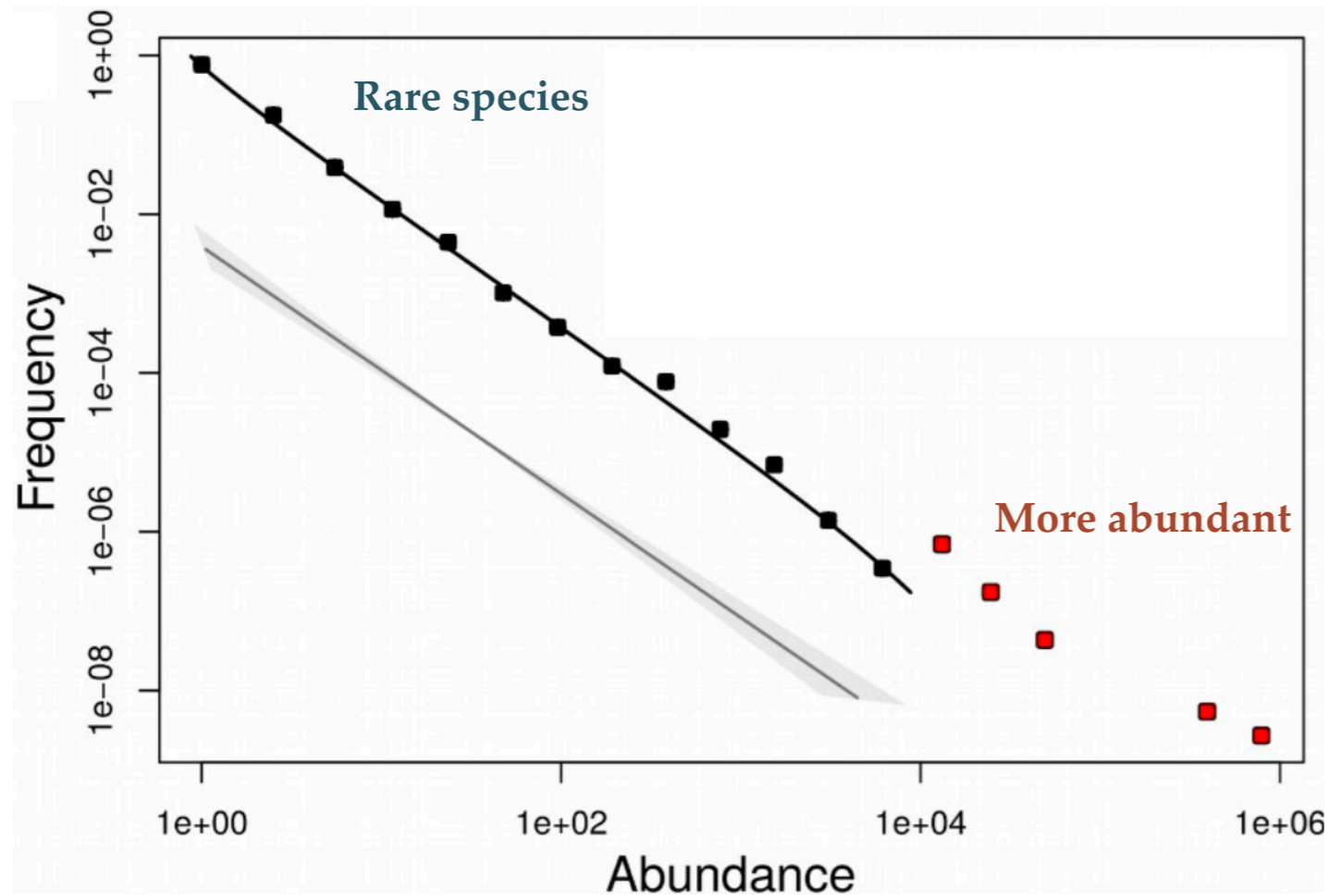# Strongly competitive interactions

Other scalings can suitably be considered in the field, e.g. $\alpha_{ij} \sim O(1)$

Chaotic phases and intermittent dynamics can appear: sudden turnovers between very low abundance periods and high abundance periods. D. A. Kessler, N. M. Shnerb, Phys. Rev. E 91 (2015)

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We specifically consider here $\mu \propto S, \quad \sigma \propto \sqrt{S}$

1) Solve the equations at finite demographic noise.

2) Then in the zero-noise regime, leading to $\longrightarrow$

$$h = \overline{N^*(z)} = \frac{\sqrt{q_0}\sigma}{1 - \sigma^2 \Delta q} \, \omega_1(\Delta)$$

$$q_0 = \overline{N^*(z)^2} = \frac{q_0 \sigma^2}{(1 - \sigma^2 \Delta q)^2} \, \omega_2(\Delta)$$

$$\Delta q = \frac{\overline{\theta(N^*(z))}}{H''_{RS}(N^*(z))} = \frac{1}{1 - \sigma^2 \Delta q} \, \omega_0(\Delta)$$

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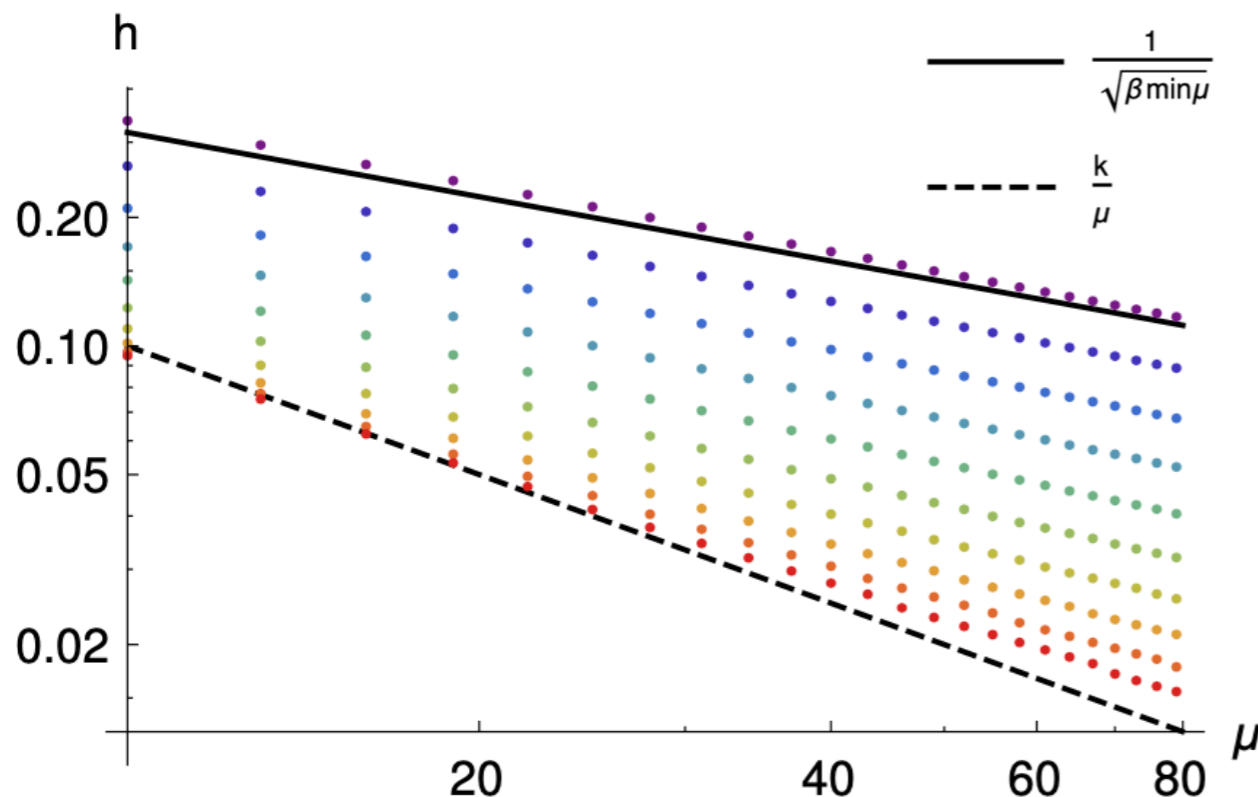
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$$\omega_i(\Delta) = \int_{-\Delta}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} (z + \Delta)^i \qquad \Delta = \frac{k - \mu h}{\sqrt{q_0}\sigma}$$
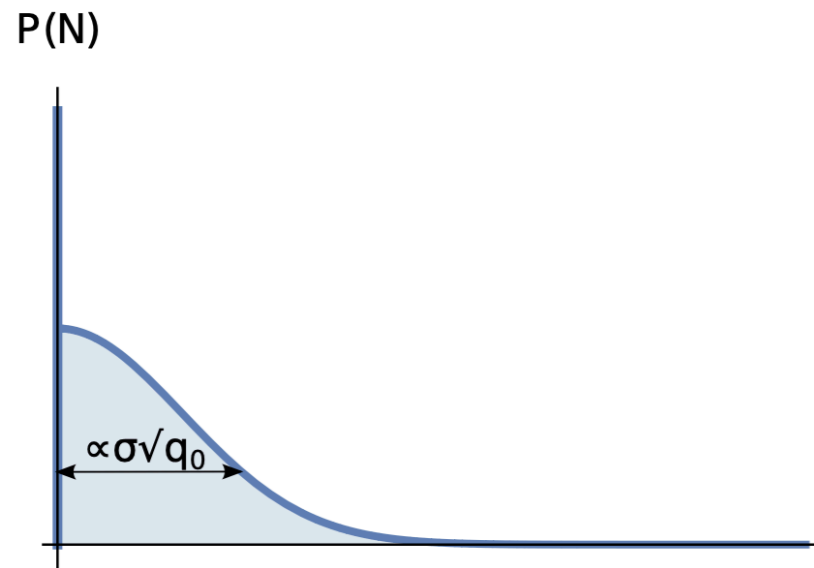

- Thermal fluctuations strongly suppress the diversity

$$\phi = \int_{-\Delta}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$
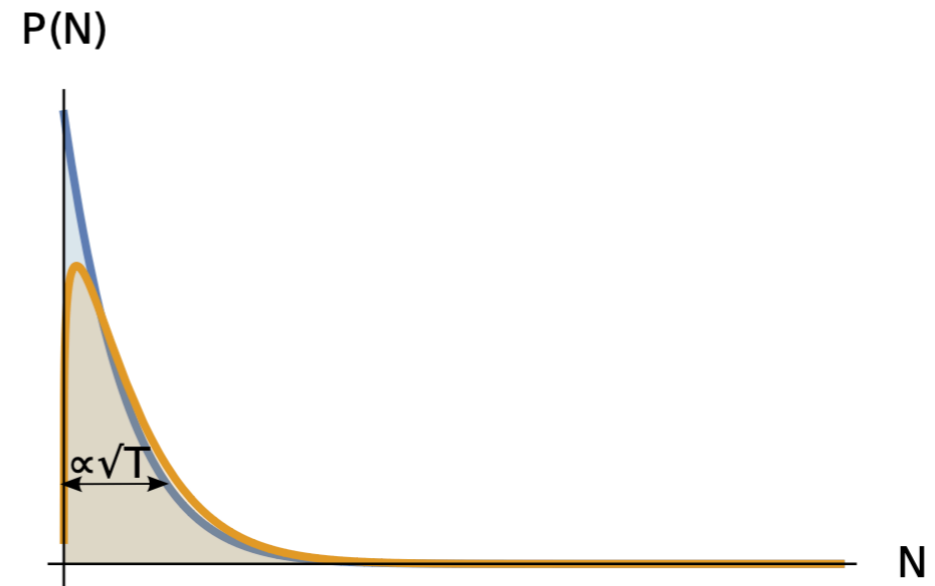
- Diversity bound saturated (T=0)

In both cases, we find out truncated Gaussian distributions.

P(N)

$\propto \sigma \sqrt{q_0}$

(a) $T = 0$

P(N)

$\propto \sqrt{T}$

N

(b) $T > 0$

- The finite width of the distribution is due to **disorder**.

- Truncated Gaussian $+ \delta$ - function.

- **The disorder is negligible**.

- Variability in the abundances is only due to thermal fluctuations.

- Still the tail of a Gaussian distribution.

- Similar distributions found in disordered systems in presence of hard constraints.

- How to go further and obtain **power-law scalings**?

De Martino, M. Marsili, I. P. Castillo, J. Stat. Mech. P04002 (2014)
M. Tikhonov, R. Monasson, Phys. Rev. Lett. 118 (2017)
S. Franz, G. Parisi, J. Phys. A: Math. Theor. 49 (2016). 23

Next step: modelling intra-specific cooperative effects
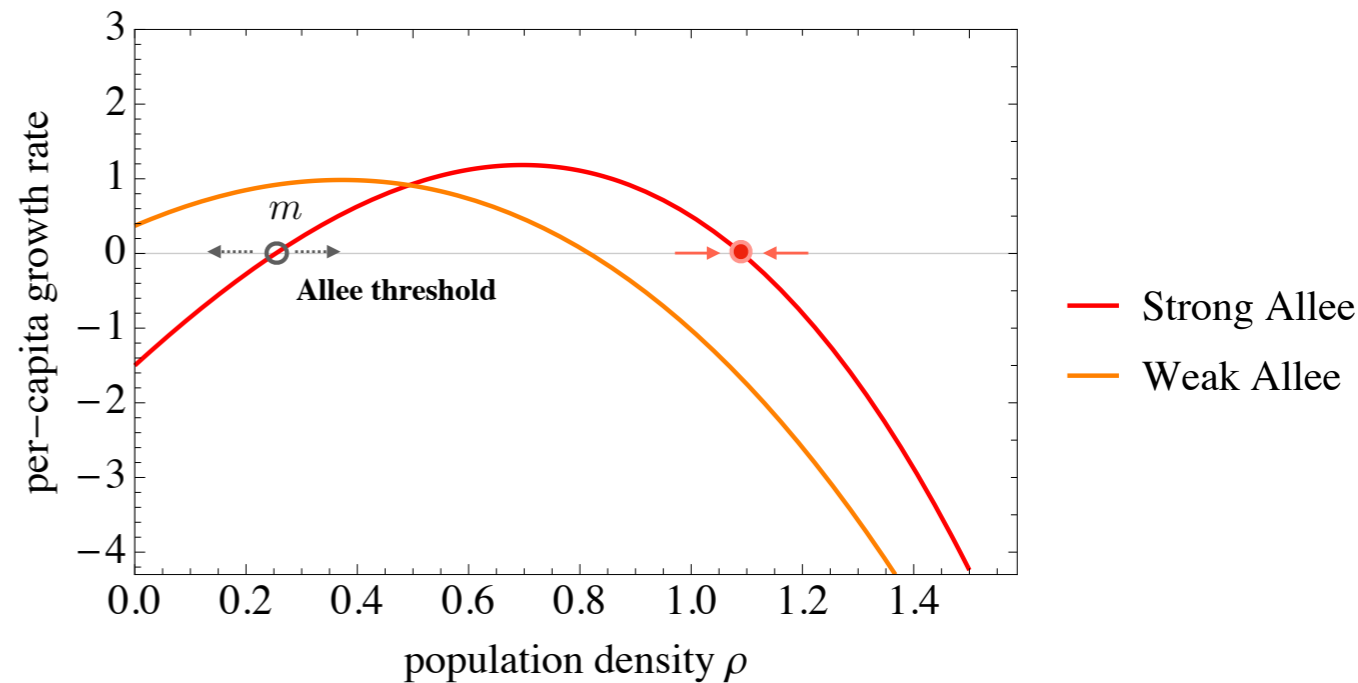
$$\frac{dN_i}{dt} = -N_i \left[\nabla_{N_i} V_i(N_i) + \sum_{j(j\neq i)} \alpha_{ij} N_j \right] + \sqrt{N_i}\eta_i(t) + \lambda_i$$

$$\langle \eta_i(t)\eta_j(t') \rangle = 2T\delta_{ij}\delta(t - t')$$

Consider a higher-order potential with finite threshold

$$V_i(N_i) = -r_i \left(1 - \frac{N_i}{K_i} \right)(N_i - m)N_i$$

Modelization of the so-called **Allee effect** = positive correlation between population density and mean individual fitness.

In many species **undercrowding** - not only competition - contributes to limiting population growth.

Empirical evidences in marine populations, reptiles and mammals.

Recently, increasing interest **in engineered biology, cancer cell evolution and epidemiology**.
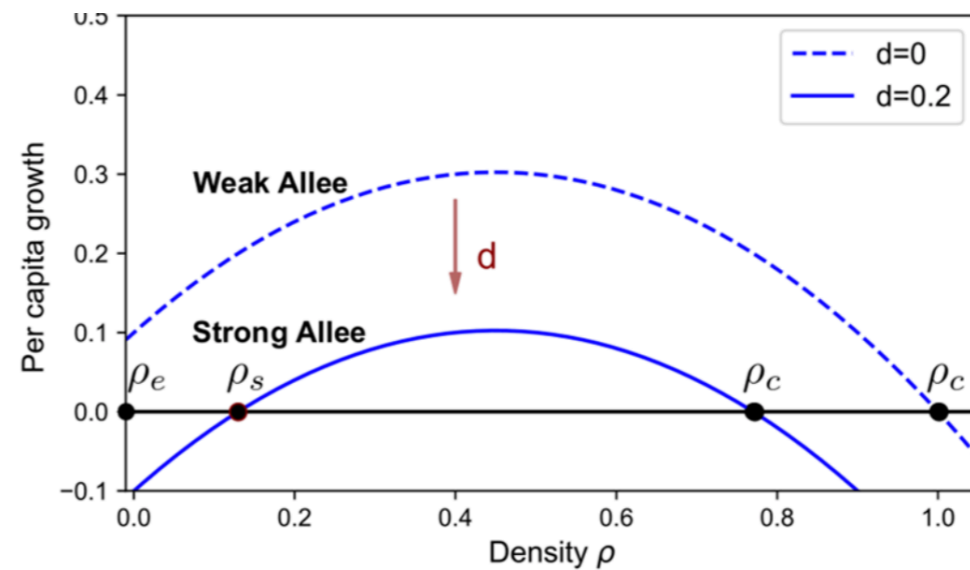
W.C. Allee, E. Bowen, Journal of Experimental Zoology (1932)
M.A. Lewis, P. Kareiva, Theoretical Population Biology (1993).

Another noteworthy example is given by microbial ecosystems.

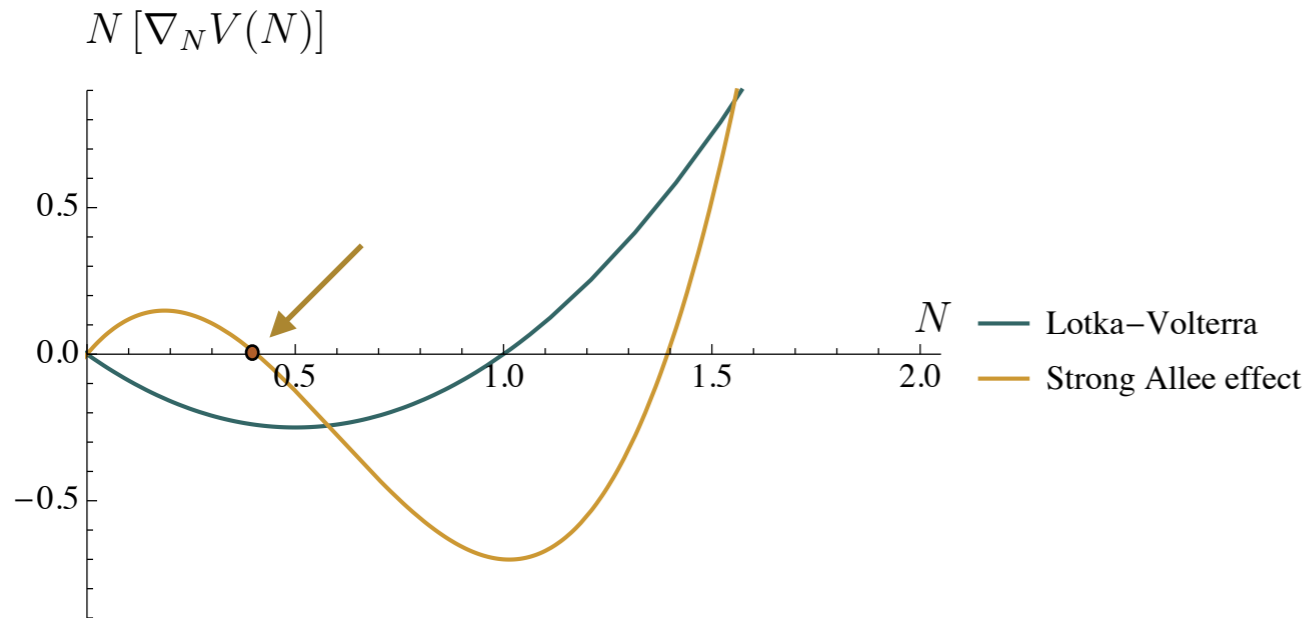Mutualism, as a result of metabolic cross-feeding, is responsible for a weak Allee effect.
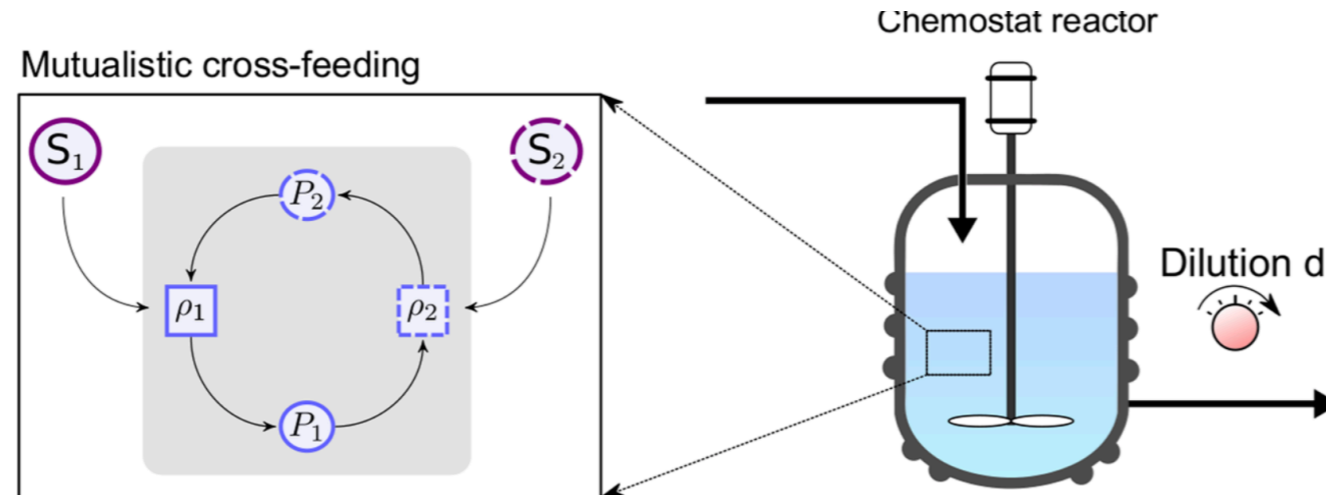
Dilution is a crucial condition for bistability.
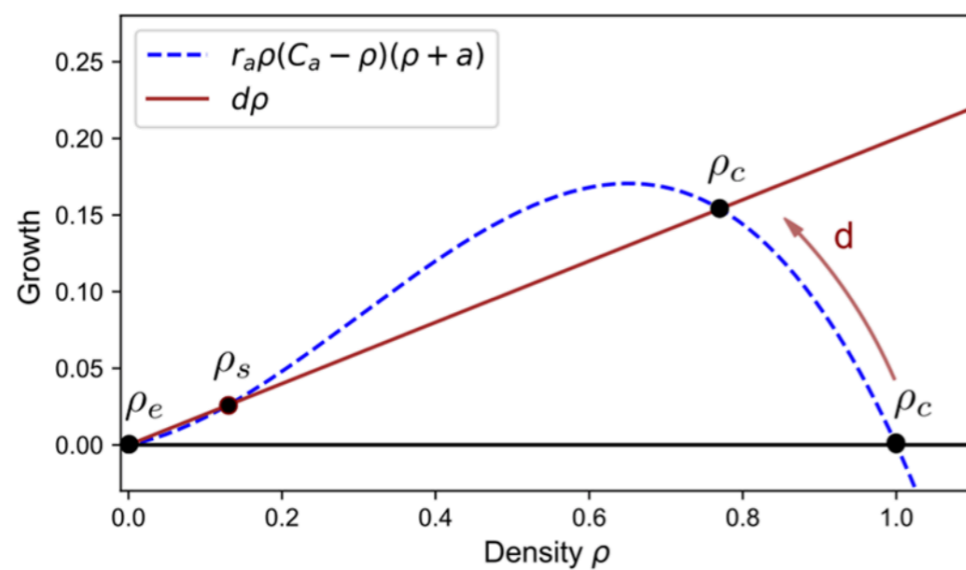
The introduction of a cubic potential complicates the analysis and leads to *a priori* unsolved competition.

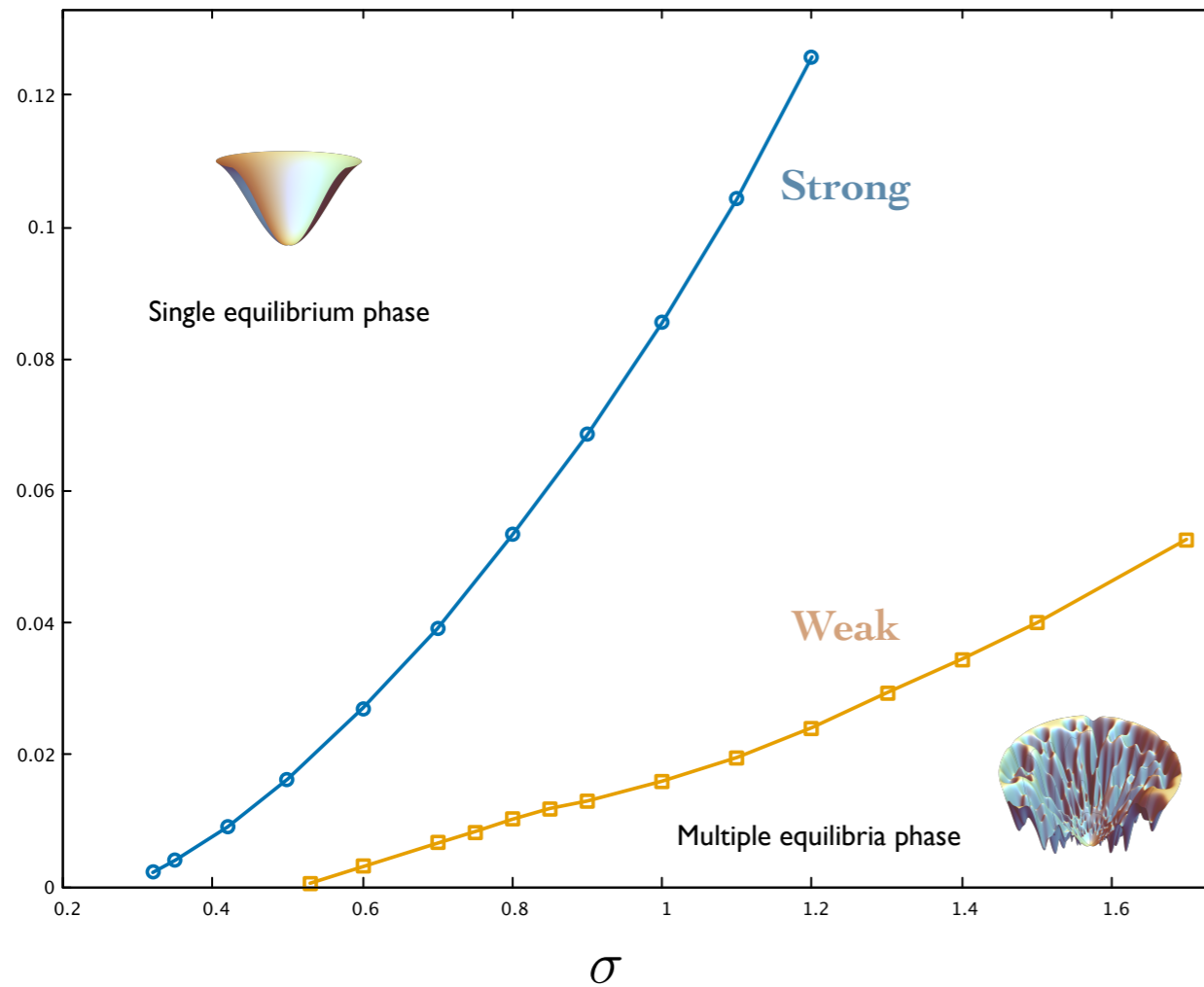
$N\left[\nabla_N V(N)\right]$

- The globally asymptotic unstable fixed point changes to a **neutral stable fixed point**;

- A strong Allee effect (positive threshold) is associated with a double-well potential;

- A weak effect (negative or zero threshold) is framed by a single-well potential.
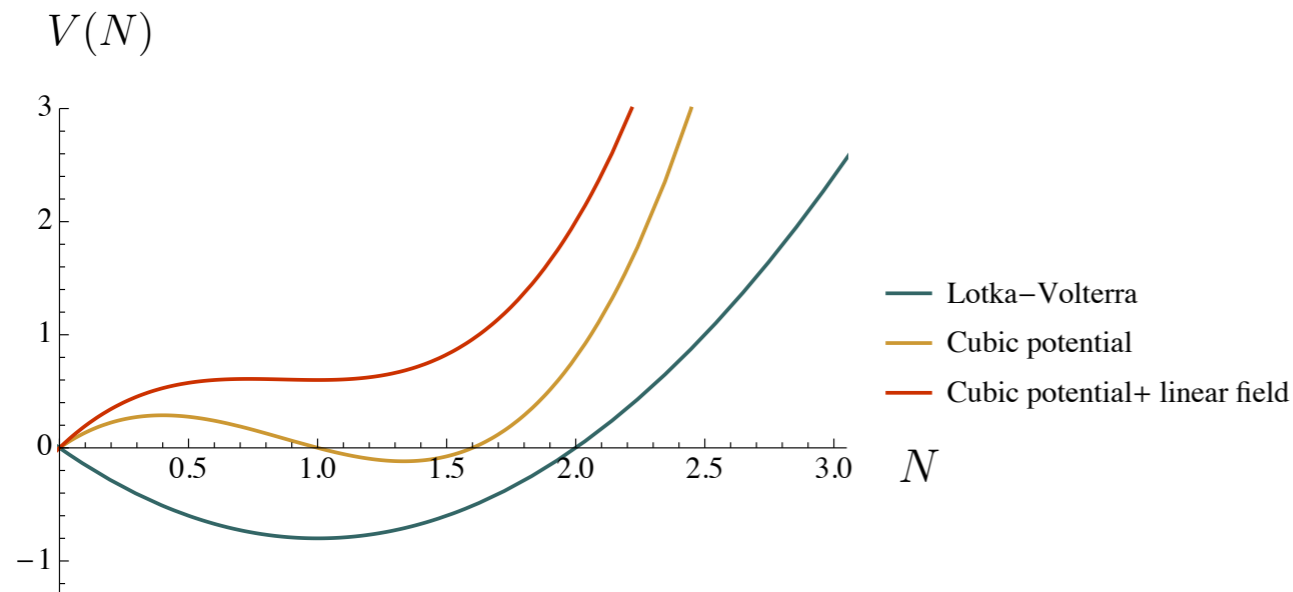
$V(N)$

- For the Allee case, only two phases take place.
  No evidence of an **amorphous Gardner**-like phase.

- **The functional response**, at variance with the LV model,
  depends on the species abundance.

The smallest eigenvalue of the stability matrix:

$$\lambda_{\mathrm{R}} = (\beta\rho\sigma)^2 \left[1 - \frac{\rho^2\sigma^2}{(-\Delta q\rho^2\sigma^2 + V''(N^*))^2} \right]$$
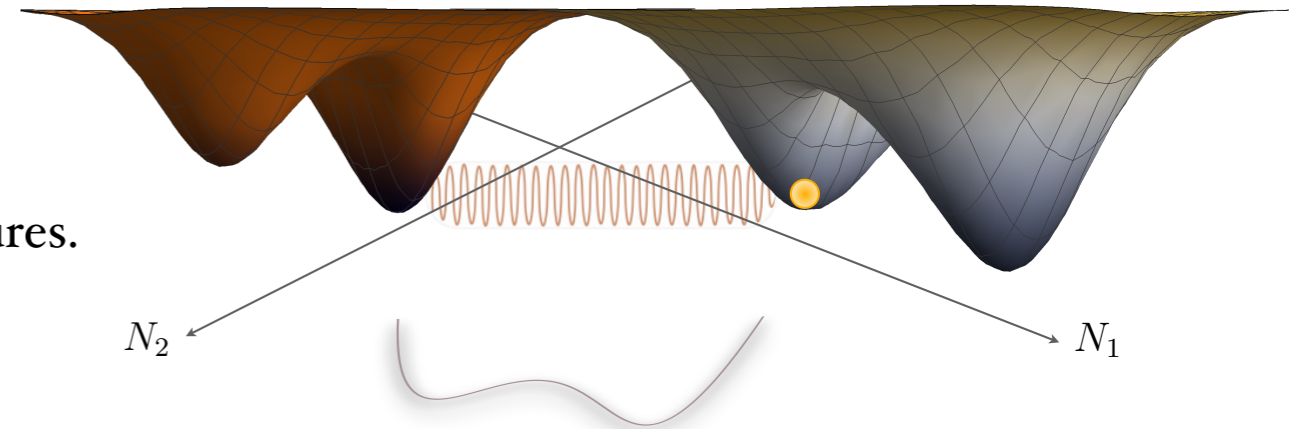
Heterogeneity   *Effective potential*

This last condition can be rephrased in terms of the probability distribution of the local curvatures of the potential:

Condition for stability (or marginal stability): $1 - (\rho\sigma)^2 \int \frac{dV''_{\mathrm{eff}} \, P(V''_{\mathrm{eff}}(N^*))}{\left(V''_{\mathrm{eff}}(N^*)\right)^2} \geq 0$

In the models of ecosystems we are considering,
a peculiar marginality condition appears.

It leads to a constraint on the fluctuations of the local curvatures.

N_2 $\qquad$ N_1

Dynamical cavity argument

$$\dot{N}_i(t) = N_i \left[-V'(N_i) - \sum_{j \neq i} \alpha_{ij} N_j + \xi_i \right]$$

External perturbation

$$N_i = N_{i \setminus k} - \frac{\partial N_i}{\partial \xi_i} \alpha_{ik} N_k$$

Cavity field

↓

Diagonal susceptibility

Model of Gaussian coupled potential

$$\mathcal{E}_{ij} = \frac{1}{2} N_i^2 V''_{\text{eff}}(N_i^*) + \frac{1}{2} N_j^{*2} V''_{\text{eff}}(N_j) + \alpha_{ij} N_i N_j$$

$$\mathbf{A} = \begin{pmatrix} V''_{\text{eff}}(N_i^*) & \alpha_{ij} \\ \alpha_{ij} & V''_{\text{eff}}(N_j^*) \end{pmatrix}$$

Energy variation due to an infinitesimal field:

$$\delta \mathcal{E} = \frac{1}{2} \sum_i \delta N_i^2 \, V''_{\text{eff}}(N_i^*)$$

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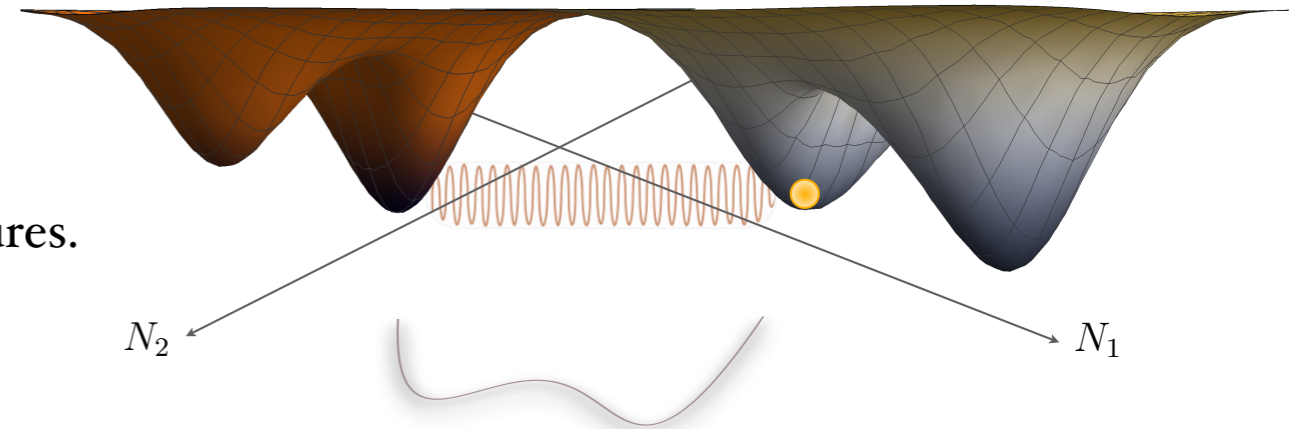
Energy variation due to an infinitesimal field:

$$\delta \mathcal{E} = \frac{1}{2} \sum_i \delta N_i^2 \, V_{\text{eff}}''(N_i^*)$$

In both cases, when looking at minimal local curvatures $V_{\text{eff}}''(N^*) \sim V_{\text{min}}''$

we can deduce that if $P(V_{\text{eff}}'') \sim |V_{\text{eff}}''|^\alpha$ $\Rightarrow$ $|V_{\text{eff}}''| \sim \dfrac{1}{N^{\frac{1}{1+\alpha}}}$ with $\alpha \geq 1$

Generalized Lotka-Volterra model:

- low demographic noise and highly heterogenous interactions:

  multiple equilibria phase (locally stable) + **amorphous Gardner-like phase** (full RSB - marginal);

- One-to-one mapping with the Random Replicant Model;

- Modelization of positive feedback loop and **intra-specific cooperative effects** (different universality class).

Conclusions and Perspectives

Generalized Lotka-Volterra model:

- low demographic noise and highly heterogenous interactions:

  multiple equilibria phase (locally stable) + **amorphous Gardner-like phase** (full RSB - marginal);

- One-to-one mapping with the Random Replicant Model;

- Modelization of positive feedback loop and **intra-specific cooperative effects** (different universality class).

Work in progress...

- Introduction of a notion of **space**: model of metacommunities coupled by migration (directed-percolation coupled processes + disorder).

  Hastings, Journal of Math. Biology (1978);
  Mobilia, Taüber, J. Stat. Phys. (2007);
  Denk, Hallatschek. PNAS (2022).

- Dynamical Mean-Field Theory with **asymmetric interactions** for:

  - Mixed MacArthur model along with the analysis of slow timescales;

  - Robust theoretical framework to capture boom-and-bust population dynamics and power laws (Tara expedition data);

  - Analysis of the volume distribution of the attractors.

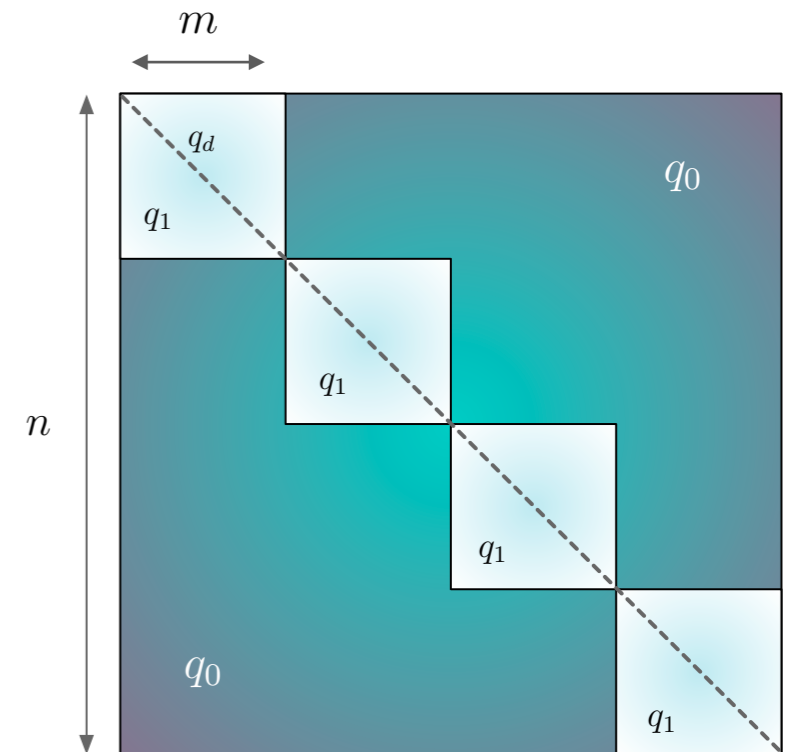*Thank you for your kind attention!*

Back-up slides

The overlap matrix $\quad Q_{ab} = \dfrac{1}{S} \sum_i \overline{\langle N_i^a N_i^b \rangle}_R$

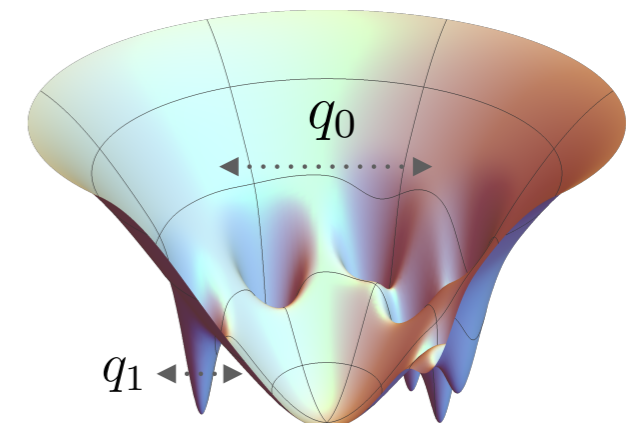must be then parametrised according to a different replica Ansatz (1-step replica symmetry breaking).

A double Gaussian integration comes into play now leading to two different averages:

$$\langle \cdot \rangle_{1\mathrm{r}} = \frac{\int_{N_c}^{\infty} dN \exp\left[-\beta H_{1\mathrm{RSB}}(N, z, t_{a_B})\right] \cdot}{\int_{N_c}^{\infty} dN \exp\left[-\beta H_{1\mathrm{RSB}}(N, z, t_{a_B})\right]} \ ,$$

$$\langle \cdot \rangle_{m\text{-}r} = \frac{\int \frac{dt_{a_B}}{\sqrt{2\pi}} e^{-\frac{t_{a_B}^2}{2}} \left(\int_{N_c}^{\infty} dN \exp\left[-\beta H_{1\mathrm{RSB}}(N, z, t_{a_B})\right] \right)^m \cdot}{\int \frac{dt_{a_B}}{\sqrt{2\pi}} e^{-\frac{t_{a_B}^2}{2}} \left(\int_{N_c}^{\infty} dN \exp\left[-\beta H_{1\mathrm{RSB}}(N, z, t_{a_B})\right] \right)^m}$$

- $q_0 = \dfrac{1}{S} \sum_i \langle N_i \rangle_\alpha \langle N_i \rangle_\beta$

- $q_1 = \dfrac{1}{S} \sum_i \langle N_i \rangle_\alpha^2$

$$h = \int \mathcal{D}z \frac{\int_{N_c}^{\infty} e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)} N}{\int_{N_c}^{\infty} dN e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)}} = \overline{\langle N \rangle}$$

Immigration

$$q_d = \int \mathcal{D}z \left(\frac{\int_{N_c}^{\infty} dN e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)} N^2}{\int_{N_c}^{\infty} dN e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)}} \right) = \overline{\langle N^2 \rangle} \ ,$$
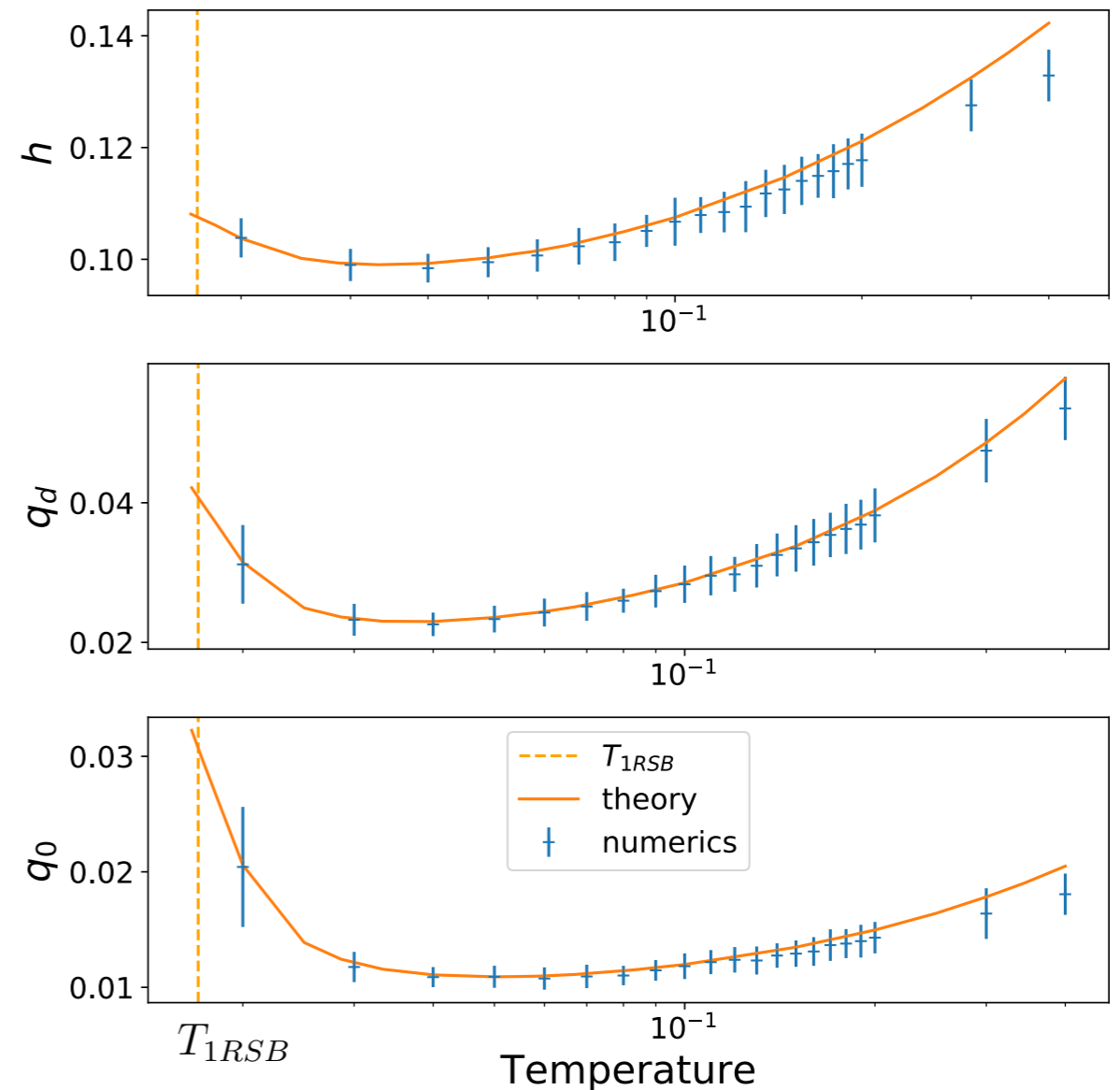
$$q_0 = \int \mathcal{D}z \left(\frac{\int_{N_c}^{\infty} dN e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)} N}{\int_{N_c}^{\infty} dN e^{-\beta H_{\text{RS}}(q_0, q_d, h, z)}} \right)^2 = \overline{\langle N \rangle^2} \ ,$$

$$h = \mathbb{E}\left[N(t)\right]$$

$$q_d = C(0) = \mathbb{E}\left[N(t)^2\right]$$

$$q_0 = \lim_{\tau \to \infty} C(\tau) = \lim_{\tau \to \infty} \mathbb{E}\left[N(t)N(t+\tau)\right]$$

Ergodicity breaking effects start to appear.

A. Altieri, F. Roy C. Cammarota, G. Biroli, Phys. Rev. Lett. **126**, 258301 (2021)

Goal: determining the number of local minima of the free energy, averaged over quenched disorder.

$$\Sigma(f) \equiv \lim_{S \to \infty} \frac{1}{S} \overline{\log \mathcal{N}(f)}$$

Log of the number of minima divided by the # of species.

$$\Sigma = m^2 \frac{d}{dm} \left(\beta F^{1\mathrm{RSB}} \right) = -m^2 \frac{d}{dm} \left(\frac{1}{n} \ln \overline{\int dq_d dq_1 dq_0 dh \; e^{S\mathcal{A}(q_d, q_1, q_0, h)}} \right)$$

$$\Sigma = \frac{m^2 \rho^2 \sigma^2 \beta^2}{4}(q_1^2 - q_0^2) + \int \mathcal{D}z \, \ln \left[\int \frac{dt_{a_B}}{\sqrt{2\pi}} e^{-\frac{t_{a_B}}{2}} A(z, t_{a_B})^m \right] - m \int \mathcal{D}z \frac{\int \frac{dt_{a_B}}{\sqrt{2\pi}} e^{-\frac{t_{a_B}}{2}} A(z, t_{a_B})^m \ln A(z, t_{a_B})}{\int \frac{dt_{a_B}}{\sqrt{2\pi}} A(z, t_{a_B})^m}$$
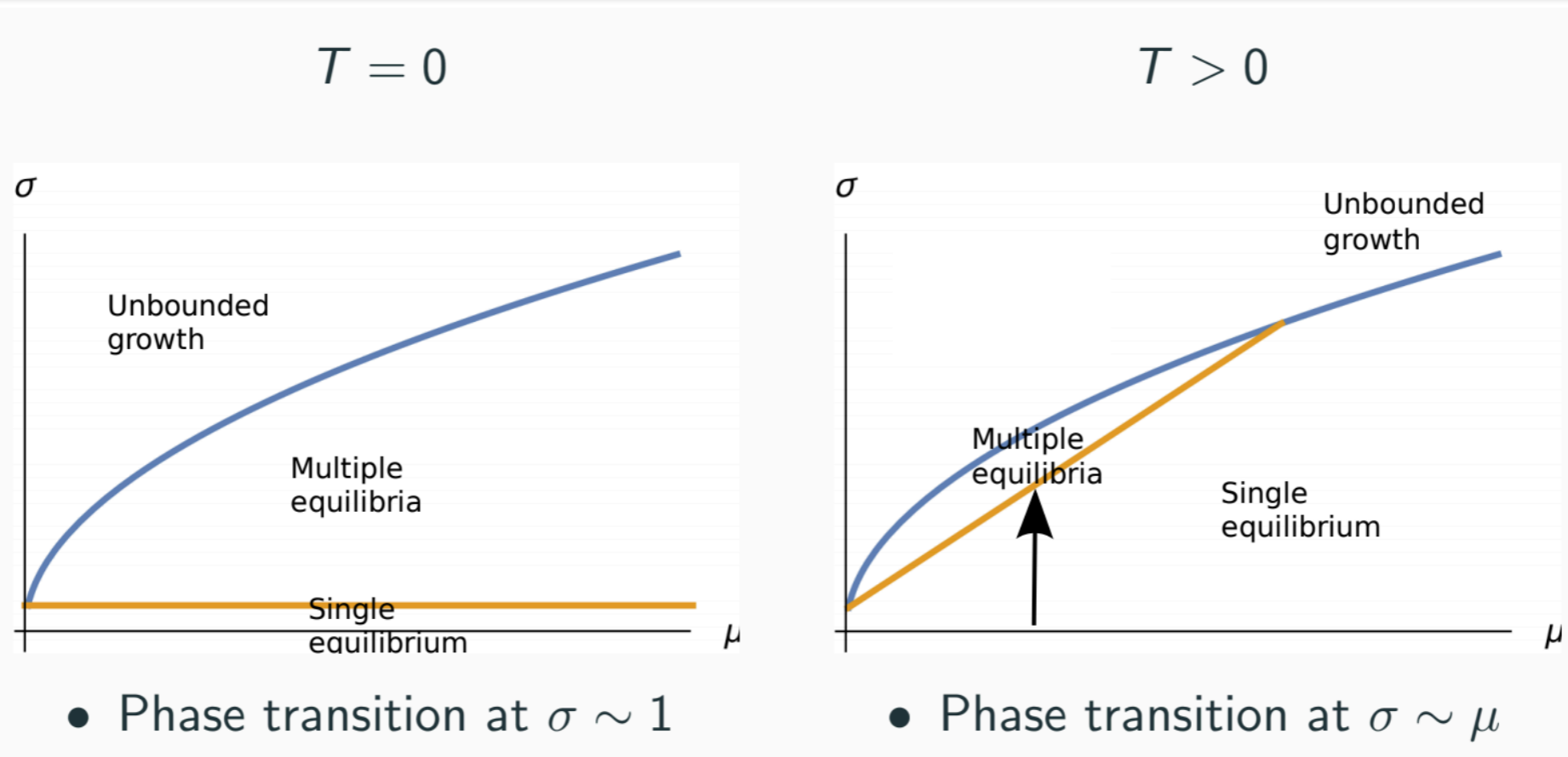
Strictly positive complexity at finite temperature (in the 1RSB phase), and compatible with previous analysis at zero temperature.

R. Monasson, Phys. Rev. Lett. **75**, 2847 (1995).

Scaling and phase diagrams in the strongly competitive case

$h = \overline{\langle N \rangle}$

$\beta = 1$

$\beta = 200$

—— $\dfrac{1}{\sqrt{\mu}}$

- - - $\dfrac{1}{\mu}$
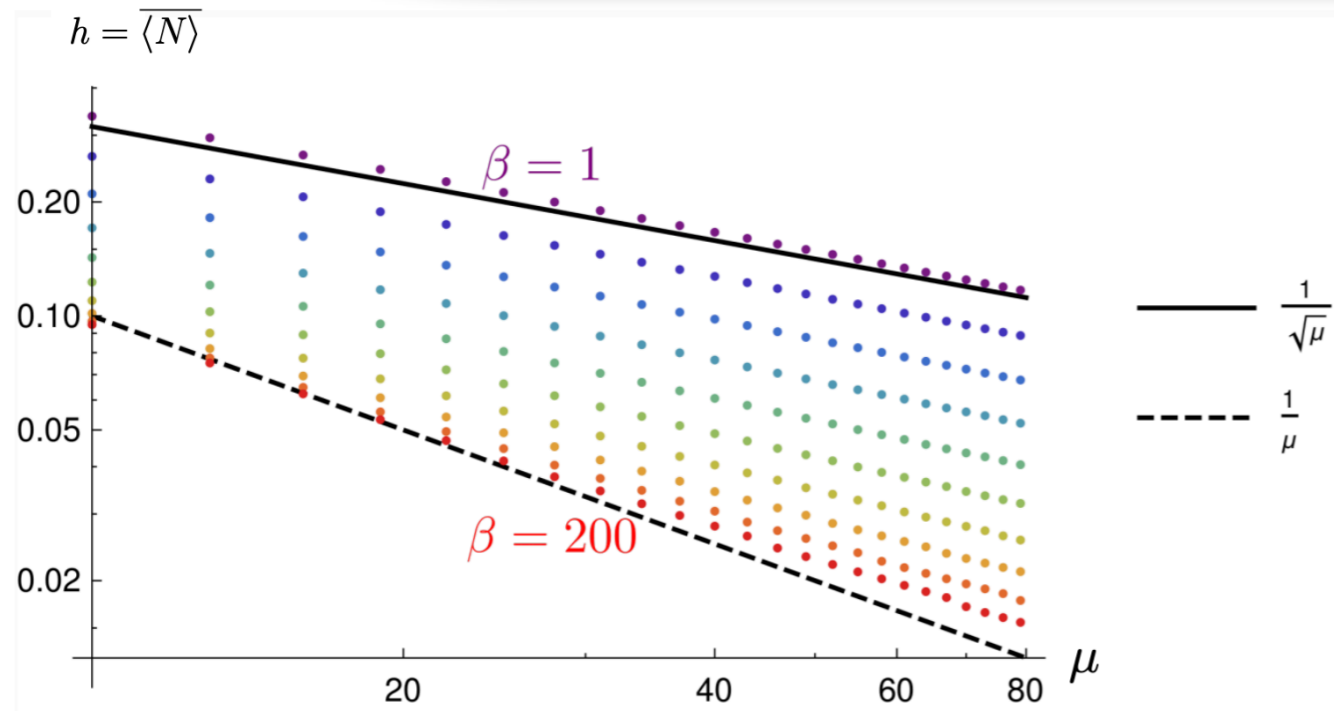
Two different scalings depending on finite or zero demographic noise regime.
Continuous crossover between the two.

Phase diagrams in the two regimes:
no difference w.r.t to the weak interaction case (T=0); dramatically affected in the T>0 case

$T = 0$

σ

Unbounded growth

Multiple equilibria

Single equilibrium

μ

- Phase transition at $\sigma \sim 1$

$T > 0$

σ

Unbounded growth

Multiple equilibria

Single equilibrium

μ

- Phase transition at $\sigma \sim \mu$